Bifurcations of vortex structures in two-dimensional flows

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Hydrodynamics at all length scales:
from high-energy to hard and soft matter
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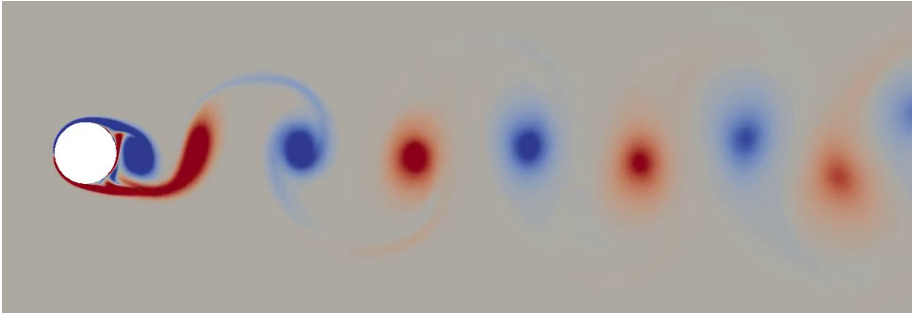
von Karman vortex street

2S wake

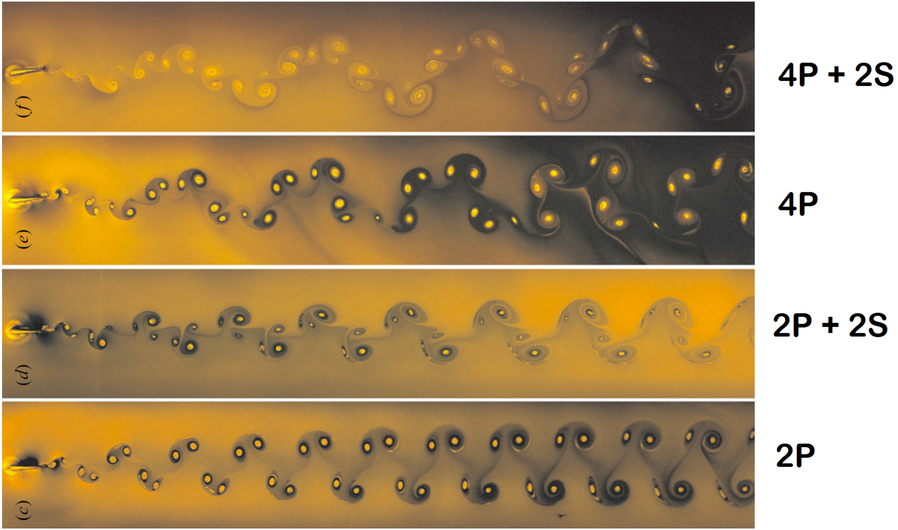


Oscillating cylinder may lead to exotic wakes

P + S wake

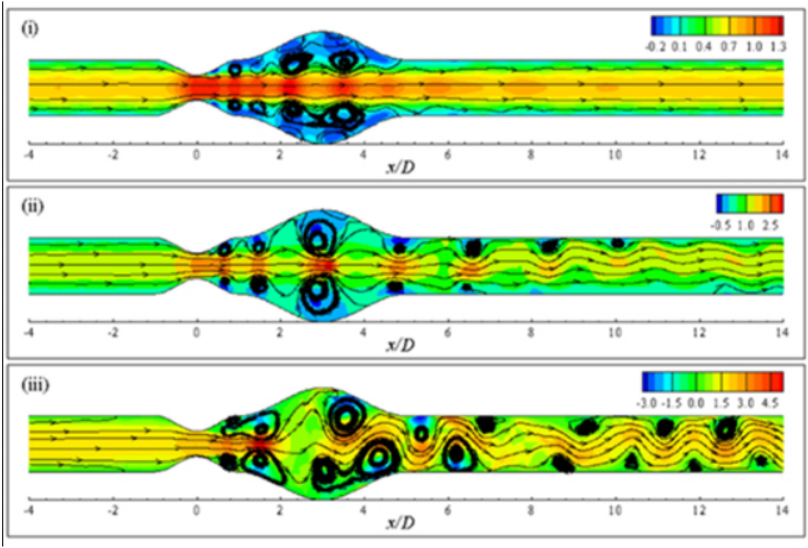
Williamson, in Ponta & Aref, J. Fluids Struct. 22(2006), 327–344

Other body shapes produce very exotic wakes



Schnipper *et al*, J. Fluid Mech. 633(2009), 411–423

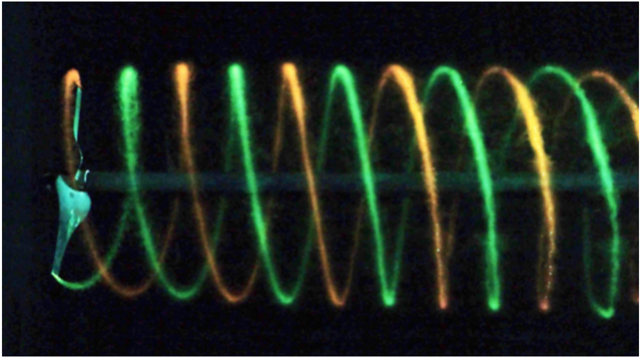
Hydrodynamics at all length scales



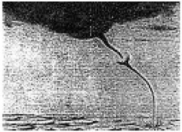
Hydrodynamics at all length scales

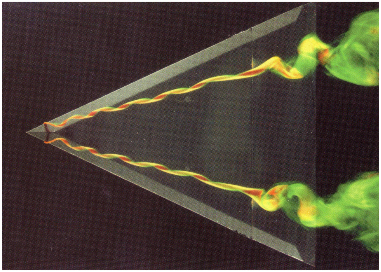
Wingtip vortex

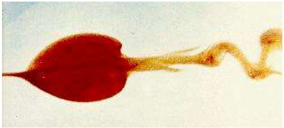
Helical vortex



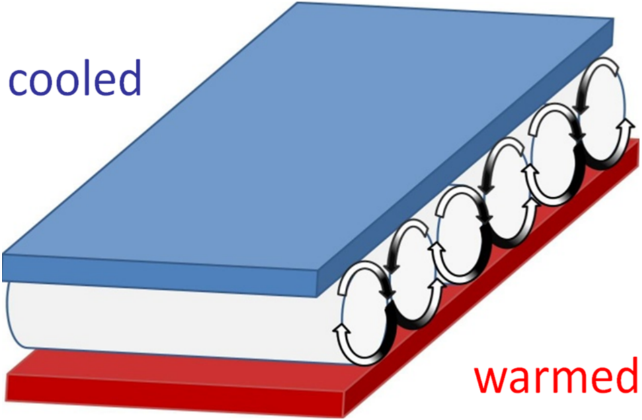
Vortex breakdown



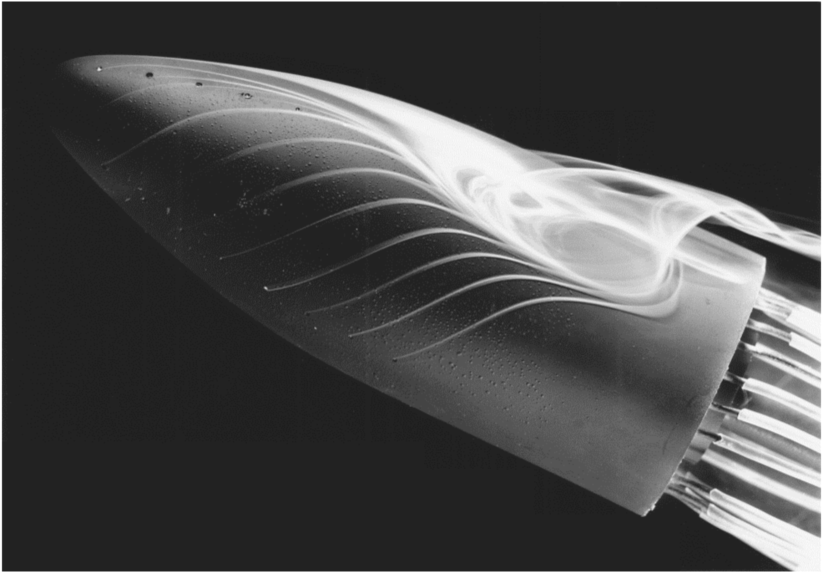




Rayleigh-Bénard convection



3D separation



What is a vortex?

From Webster

vortex noun

vor·tex | \ ˈvȯr-ˌteks \

plural **vortices** \ ˈvȯr-tə-ˌsēz \ *also* **vortexes** \ ˈvȯr-ˌtek-səz \

Definition of *vortex*

1 : something that resembles a whirlpool

// the hellish *vortex* of battle

— *Time*

2 **a** : a mass of fluid (such as a liquid) with a whirling or circular motion that tends to form a cavity or vacuum in the center of the circle and to draw toward this cavity or vacuum bodies subject to its action

especially **:** WHIRLPOOL, EDDY

b : a region within a body of fluid in which the fluid elements have an angular velocity

The notion of a vortex is so widely used in fluid dynamics that few pause to examine what the word strictly means. Those who do take a closer look quickly realize the difficulty of defining vortices unambiguously.

G. Haller, *An objective definition of a vortex*, JFM 2005

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Considerable confusion surrounds the longstanding question of what constitutes a vortex, especially in a turbulent flow. This question, frequently misunderstood as academic, has recently acquired particular significance since coherent structures (CS) in turbulent flows are now commonly regarded as vortices

J. Jeong & F. Hussain, *On the identification of a vortex*, JFM 1995

28 definitions of a vortex

Y. Zhang et al. *Renewable and Sustainable Energy Reviews 81 (2018) 1269–1285*

Table 2
Classification of the existing vortex identification methods in the literature.

Author	Year	Method	Region/Line	Invariant	Local/Global	2D/3D	Objective
Hunt et al. [17]	1988	Q-criterion	Region	Galilean	Local	3D	No
Hunt et al. [17]	1988	Maximum Q	Line	Galilean	Local	3D	No
Chong et al. [18]	1990	$\triangle$-criterion	Region	Galilean	Local	3D	No
Jeong and Hussain [19]	1995	λ_2-criterion	Region	Galilean	Local	3D	No
Jeong and Hussain [19]	1995	Minimum λ_2	Line	Galilean	Local	3D	No
Zhou et al. [20]	1999	Swirling Strength	Region	Galilean	Local	3D	No
Cucitore et al. [21]	1999	Enhanced Swirling Strength	Region	Galilean	Gobal	3D	No
Miliou et al. [22]	2005	Cut-off value λ_2	Region	Galilean	Local	3D	No
Haller [23]	2005	Mz	Region	Lagrangian	Local	3D	Yes
Green et al. [24]	2007	Lyapunov Exponent	Region	Lagrangian	Local	3D	Yes
Fuchs et al. [25]	2008	Delocalized unsteady vortex	Region	Lagrangian	Gobal	3D	No
Gűnther et al. [26]	2016	Rotation invariance	Region	Rotating	Local	3D	No
Berdahl and Thompson [27]	1993	Swirl Parameter	Region	Not	Local	3D	No
Banks and Singer [28]	1995	Predictor-Corrector	Line	Not	Local	3D	No
Cucitore et al. [21]	1999	R-definition	Line	Not	Gobal	3D	No
Lugt [29]	1999	Stream lines	Line	Not	Global	3D	No
Lugt [29]	1999	Path lines	Line	Not	Global	3D	No
Weinkauf and Theisel [30]	2010	Streak lines	line	Not	Global	3D	No
Spalart [31]	1988	Vorticity magnitude	Region	Not	Local	3D	No
Spalart [31]	1988	Vorticity lines	Line	Not	Gocal	3D	No
Kline and Robinson [32]	1990	Pressure iso-surface	Region	Not	Global	3D	No
Kline and Robinson [32]	1990	Pressure minima	Line	Not	Local	2D	No
Degani et al. [33]	1990	Helicity	Line	Not	Local	3D	No
Sujudi and Haimes [34]	1995	Eigenvector	Line	Not	Local	3D	No
Roth and Peikert [35]	1998	Parallel Vectors	Line	Not	Local	3D	No
Holmen [16]	2012	Velocity components	Region	Not	Local	3D	No
Wang and Li [36]	2014	Rotation index-based	Region	Not	Local	2D	No
Dong et al. [37]	2016	Combing λ_2 and vortex filaments	Line	Not	Local	3D	No

… further definitions appear continuously.

Overview of rest of talk

1. Streamlines, pathlines, streaklines
2. Vortex definition # 1: Closed streamlines
3. Vorticity
   - Point vortices
   - Vortex definition # 2: Extremum of vorticity
   - The onset of vortex dynamics in the cylinder wake
4. Vortex definition # 3: The Q-criterion
   - Q-vortices in boundary layer eruption
5. Summary

The setting

- The following takes place in the continuum limit – a smooth macroscopic velocity field is assumed and is all that is needed.

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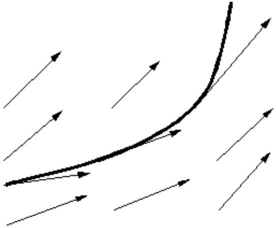
- The specific constitutive properties of the fluid is of secondary importance – special results for Newtonian fluids will be presented.

For a time-dependent velocity field $\mathbf{v}(\mathbf{x}, t)$:
The *(instantaneous) streamlines at* $t = t_0$ are the solution curves to

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t_0)$$



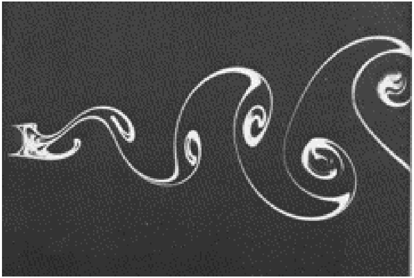
The *pathlines* are the solutions to

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x},t).$$

If dye is fed from a point $\mathbf{x}_0$ a *streakline* appears in the flow. if the pathline which fulfills the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is denoted $\mathbf{x}(t_0,t)$ the streakline at time t is the curve

$$t_0 \to \mathbf{x}(t_0,t), \quad t_0 \in [t_s,t],$$

where t_s is the time the experiment (or dye release) is started.



- If the flow is steady, $\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{x})$, streamlines, pathlines and streaklines coincide.
- In two-dimensional incompressible flow there is a *streamfunction* $\psi(\mathbf{x},t)$ such that

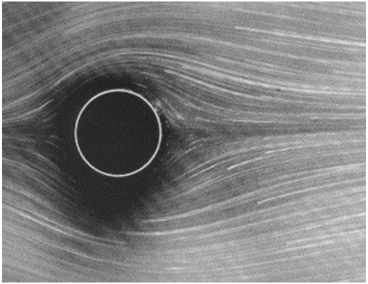
$$\mathbf{v} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\psi = \begin{pmatrix} \frac{\partial\psi}{\partial y} \\ -\frac{\partial\psi}{\partial x} \end{pmatrix}.$$

The streamlines are the level curves of ψ.

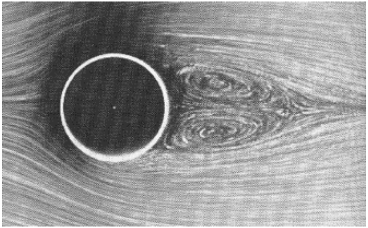
Vortex defintion # 1

A vortex in a 2D flow is a region with closed streamlines.

$Re = 1.54$



$Re = 26$



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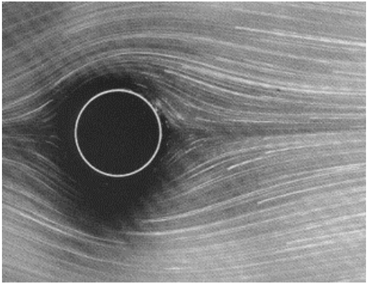

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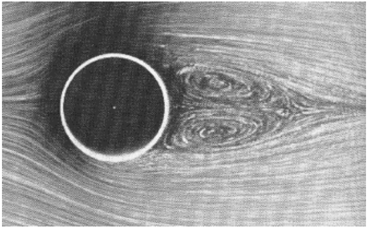


Not Galilean invariant — two observers moving with a constant relative speed will not find the same streamline structure.

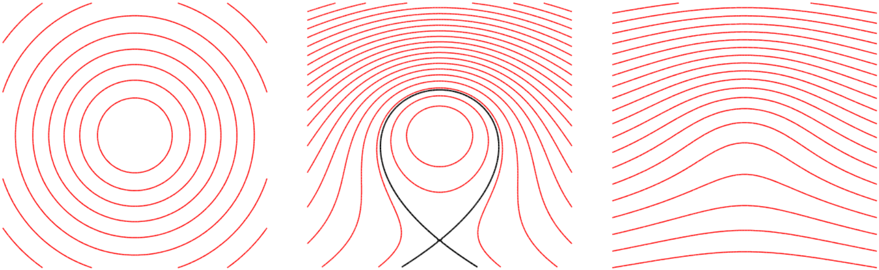
Gaussian vortex in a background flow $\mathbf{U}$

Vorticity and streamfunction for a Gaussian vortex

$$\omega = e^{-r^2}, \qquad \psi = -\frac{1}{4}\left(\ln(r^2) + \int_1^\infty \frac{e^{-ar^2}}{a}\, da\right), \qquad r^2 = x^2 + y^2$$

Streamlines

$\mathbf{U} = \mathbf{0}$ $\qquad$ $\mathbf{U} = \begin{pmatrix} -0.2 \\ 0 \end{pmatrix}$ $\qquad$ $\mathbf{U} = \begin{pmatrix} -0.4 \\ 0 \end{pmatrix}$



Streamline structure depends on the velocity of the observer – only meaningful where there is a distinguished coordinate system such as in steady flows.

Vorticity in two dimensions

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Navier-Stokes equation

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v}, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

Take the curl — vorticity transport equation

$$\frac{D\omega}{Dt} = \nu \Delta \omega$$

Ideal fluids ($\nu = 0$)
Vorticity is frozen in the fluid.
A *point vortex* of circulation Γ centered at $\mathbf{x}_0$, $\omega = \Gamma/(2\pi)\delta(\mathbf{x} - \mathbf{x}_0)$ induces a velocity field

$$\mathbf{v} = \frac{\Gamma}{2\pi} \frac{\widehat{\mathbf{x} - \mathbf{x}_0}}{|\mathbf{x} - \mathbf{x}_0|^2}$$

Ideal fluids cont.

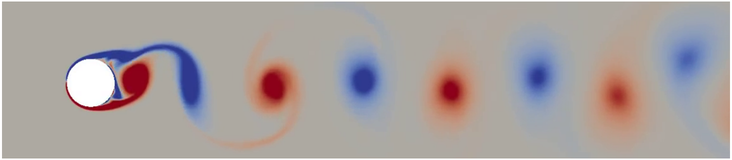
N point vortices placed at $\mathbf{x}_\alpha, \alpha = 1, \ldots, N$ induce a velocity field

$$\mathbf{v} = \sum_{\alpha=1}^{N} \frac{\Gamma_\alpha}{2\pi} \frac{\widehat{\mathbf{x} - \mathbf{x}_\alpha}}{|\mathbf{x} - \mathbf{x}_\alpha|^2}$$

Each vortex is a material point and moves in the velocity field from the other vortices

$$\frac{d\mathbf{x}_\beta}{dt} = \sum_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{N} \frac{\Gamma_\alpha}{2\pi} \frac{\widehat{\mathbf{x}_\beta - \mathbf{x}_\alpha}}{|\mathbf{x}_\beta - \mathbf{x}_\alpha|^2}, \quad \beta = 1, \ldots, N.$$

Point vortices successfully used to model cylinder wakes (von Kármán, 1912)



How to generalize to real viscous flows where vorticity diffuses, $\frac{D\omega}{Dt} = \nu \Delta \omega$

A vortex is a region of concentrated vorticity.

Vortex definition #2

A vortex is a region of concentrated vorticity.

Vorticity is Galilean invariant.

Generalizing point vortices

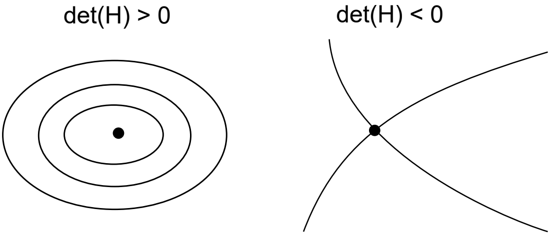
A *feature point* of a vortex is a local extremum of vorticity,

$$\partial_x \omega = \partial_y \omega = 0,$$

$$\det(H) > 0, \quad H = \begin{pmatrix} \partial_{xx}\omega & \partial_{xy}\omega \\ \partial_{xy}\omega & \partial_{yy}\omega \end{pmatrix}.$$

If $\det(H) < 0$, the critical point of ω is a saddle.

Iso-curves of ω near a critical point



Motion of critical points of vorticity (extrema and saddles)

A critical point $(x(t),y(t))$ of vorticity fulfills

$$\partial_x \omega(x(t),y(t),t) = 0, \;\; \partial_y \omega(x(t),y(t),t) = 0$$

Implicit differentiation yields equations of motion

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -H^{-1} \begin{pmatrix} \partial_{xt}\omega \\ \partial_{yt}\omega \end{pmatrix} = \begin{pmatrix} \dfrac{\partial_{xy}\omega\partial_{yt}\omega - \partial_{yy}\omega\partial_{xt}\omega}{\det(H)} \\ \dfrac{\partial_{xy}\omega\partial_{xt}\omega - \partial_{xx}\omega\partial_{yt}\omega}{\det(H)} \end{pmatrix}$$

Vortices are created or destroyed when $\det(H) = 0$

Cusp or saddle-center bifurcation of vortices

Theorem Assume the Hessian H has zero as a simple eigenvalue at a critical point at $(x,y,t) = (0,0,0)$, and choose the coordinate system such that

$$H(0,0,0) = H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \partial_{yy}\omega_0 \end{pmatrix}$$

Assume the non-degeneracy conditions

$$A = \partial_{yy}\omega_0 \neq 0, \quad B = \partial_{xt}\omega_0 \neq 0, \quad C = \partial_{xxx}\omega_0 \neq 0.$$

Then there are critical points of vorticity given by

$$x(t) = \pm\sqrt{-\frac{2B}{C}t + \mathcal{O}(t)}, \quad y(t) = \left(-\frac{1}{A}\partial_{yt}\omega_0 + \frac{B}{AC}\partial_{xxy}\omega_0\right) t + \mathcal{O}(t^{3/2})$$

If $B/C > 0$ the two critical points exist for $t < 0$ and merge and disappear at the origin at $t = 0$. If $B/C < 0$ the points are created at $t = 0$ and exist for $t > 0$. In both cases, one of the critical points is a saddle, the other is an extremum.

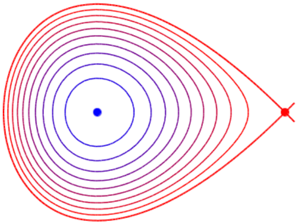
Cusp bifurcation example

$$\omega = t(y - x) + y^2 - \frac{1}{3}x^3, \qquad \partial_x \omega = -t - x^2, \quad \partial_y \omega = t + 2y.$$

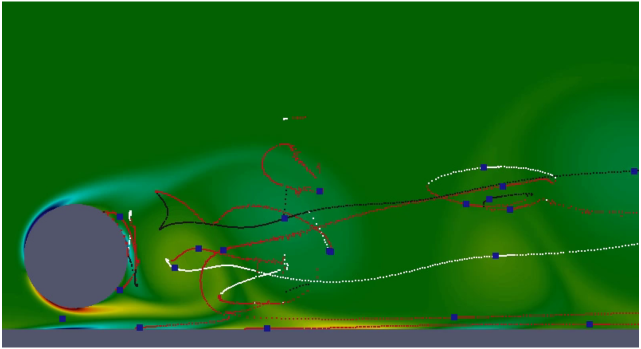
Critical points

$$x = \pm\sqrt{-t}, \quad y = -\frac{1}{2}t.$$

Easy to check that the assumptions of the theorem are fulfilled at $(x,y,t) = (0,0,0)$.



Wake of cylinder close to wall



Rasmus Ellebæk Christiansen, Master's Thesis, 2013.

The role of the vorticity transport equation

$$\partial_t \omega + (\mathbf{v} \cdot \nabla)\omega = \nu \Delta \omega$$

In the regular case, the equations of motion for the critical points of vorticity become

$$\dot{x} = u - \nu \frac{\partial_{yy}\omega \Delta \partial_x \omega - \partial_{xy}\omega \Delta \partial_y \omega}{\det(H)}$$
$$\dot{y} = v - \nu \frac{\partial_{xx}\omega \Delta \partial_y \omega - \partial_{xy}\omega \Delta \partial_x \omega}{\det(H)}$$

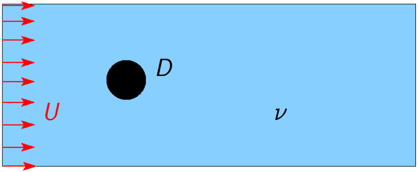
For the cusp bifurcation we get

$$B = \partial_{xt}\omega_0 = \nu \Delta \partial_x \omega_0, \qquad \frac{x(t)^2}{\nu t} = -2\left(1 + \frac{\partial_{xyy}\omega_0}{C}\right) + \mathcal{O}(t)$$

$$y(t) = \left(v + \nu \frac{\partial_{xxy}\omega_0 \partial_{xyy}\omega_0 - \partial_{xxx}\omega_0 \partial_{yyy}\omega_0}{A}\right) t + \mathcal{O}(t^2)$$

The onset of vortex dynamics in the cylinder wake

Heil, Rosso, Hazel, MB, J Fluid Mech. (2017), vol. 812, pp. 199–221.



- Flow is steady and symmetric at modest Reynolds numbers.
- Steady flow becomes unstable at

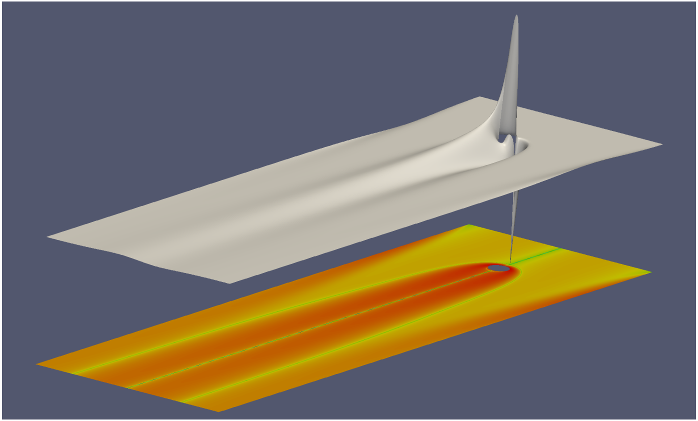
$$Re_{\mathrm{crit}} = \frac{U_{\mathrm{crit}} D}{\nu} \approx 46$$

via a symmetry-breaking, super-critical Hopf-bifurcation.

- Instability leads to formation of "Karman vortex street" via periodic shedding of vortices with a characteristic frequency.

Vorticity field pre- and post-Hopf bifurcation

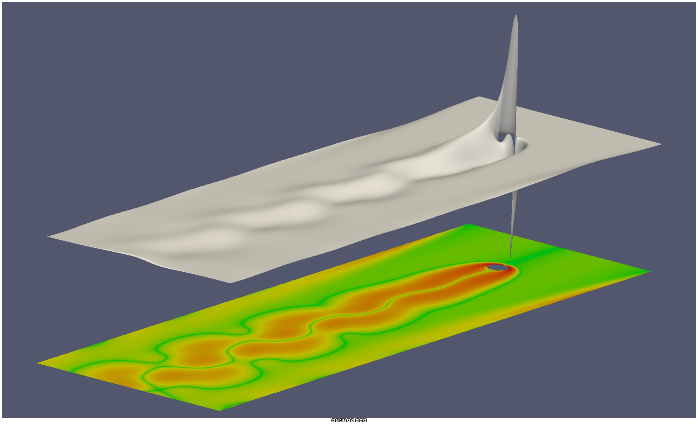
- **Before Hopf bifurcation:** Vorticity is generated on no-slip boundaries and then advected downstream; diffusion spreads out the profile as $x \to \infty$. Flow is symmetric about $y = 0$.



"Carpet plot" of vorticity, $z = \omega(x,y)$, above logarithmic colour contours of $|\omega(x,y)|$.

Vorticity field pre- and post-Hopf bifurcation

- **After Hopf bifurcation:** Time-periodic, asymmetric flow. Vorticity field is advected downstream [Karman vortex street].



"Carpet plot" of vorticity, $z = \omega(x,y,t)$, above logarithmic colour contours of $|\omega(x,y,t)|$.

Fluid dynamics near the Hopf bifurcation

Difficult to simulate close to bifurcation due to long transients

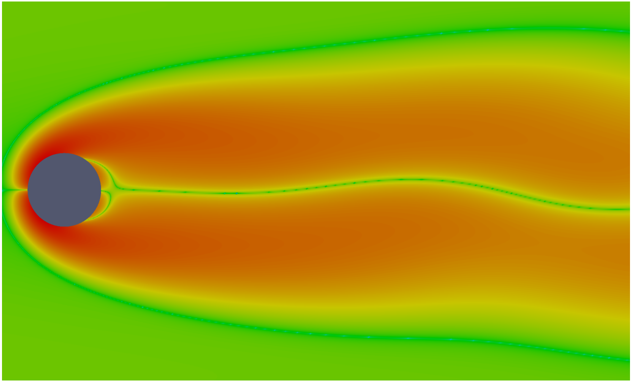
Theory: Flow close to the Hopf bifurcation is well approximated by

$$\mathbf{v}(x,y,t;Re) \approx \mathbf{v}(x,y;Re_{\mathrm{crit}}) + \varepsilon\ \widehat{\mathbf{v}}(x,y)\ e^{\mathrm{i}\Omega t}$$

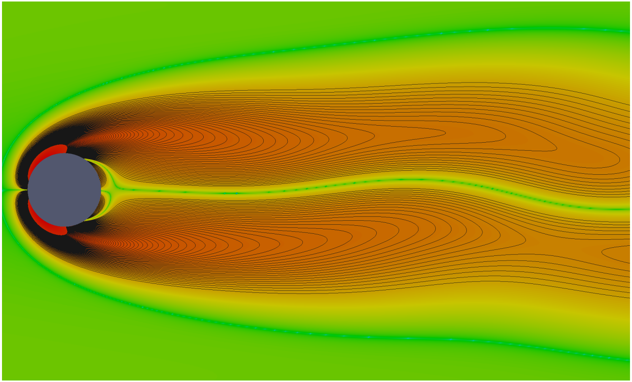
where

- $\mathbf{v}(x,y;Re_{\mathrm{crit}})$ is the steady flow at the Hopf bifurcation
- $\widehat{\mathbf{v}}(x,y)$ is a critical eigenfunction of the linearized problem at the Hopf bifurcation
- $\mathrm{i}\Omega$ is the corresponding critical eigenvalue
- $\varepsilon \sim (Re - Re_{\mathrm{crit}})^{1/2}$ is a proxy for the excess Reynolds number (above the critical value).

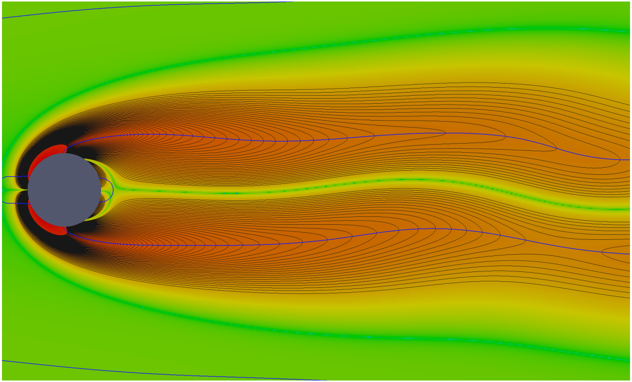
How (and where) are the extrema in the vorticity generated?



- Logarithmic colour contours of $|\omega(x,y,t)|$.



- Logarithmic colour contours of $|\omega(x,y,t)|$.
- Iso-lines of $\omega(x,y,t)$ (black lines).



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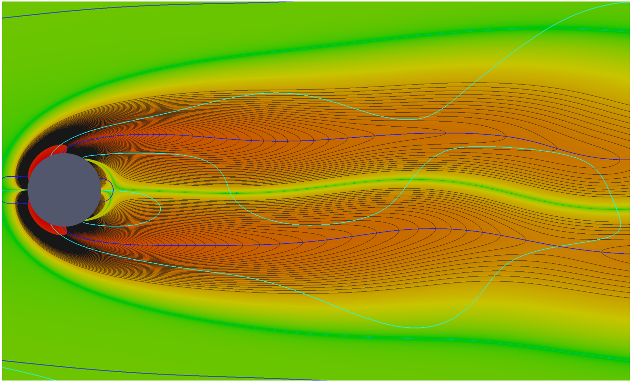

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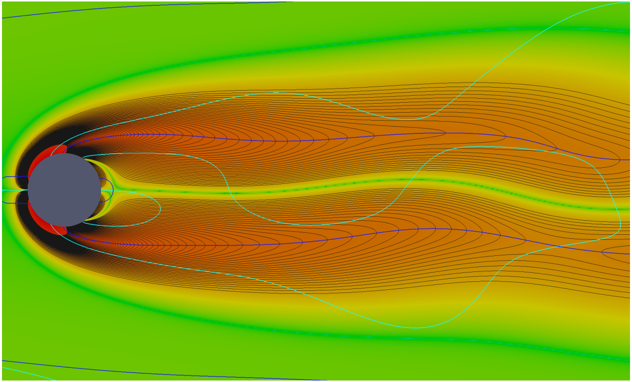

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- Intersections = critical points of vorticity field.

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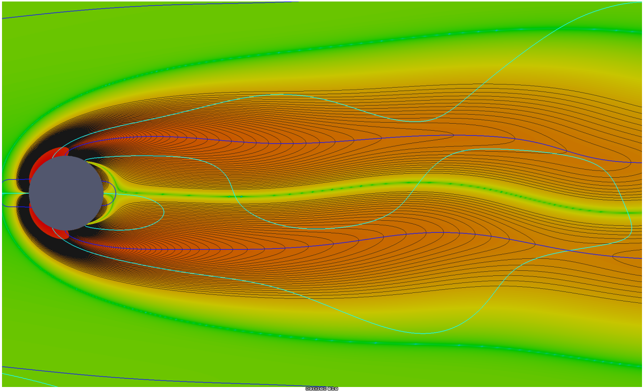

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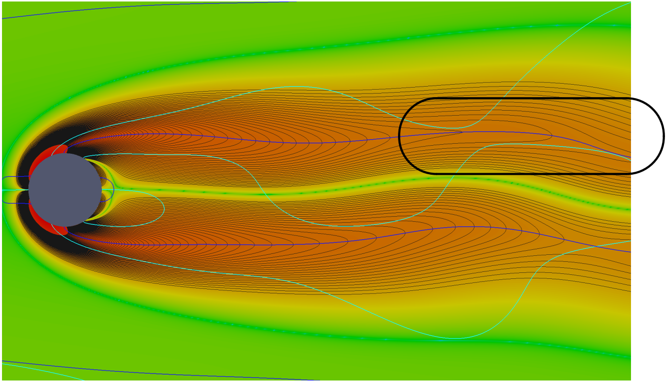

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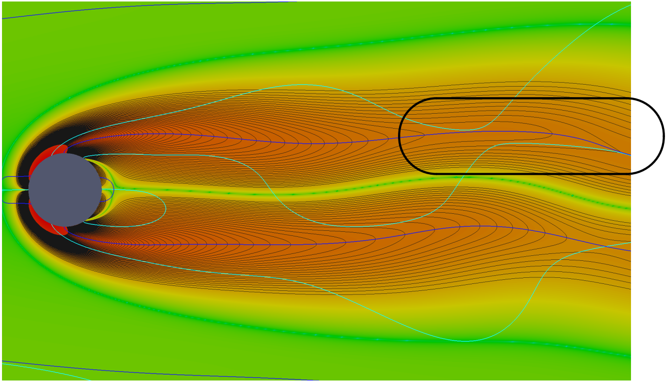

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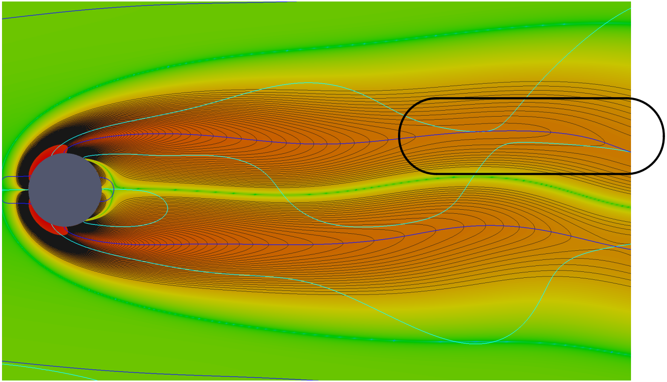

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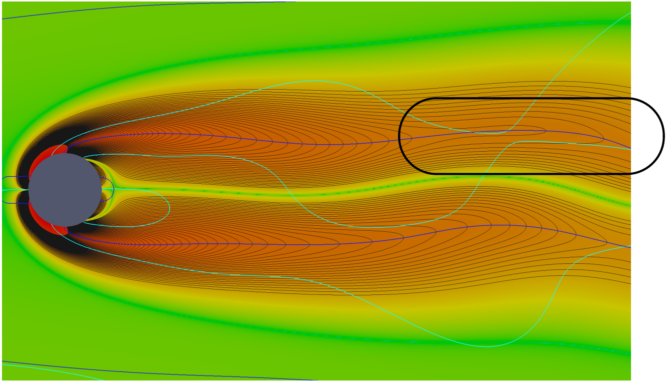

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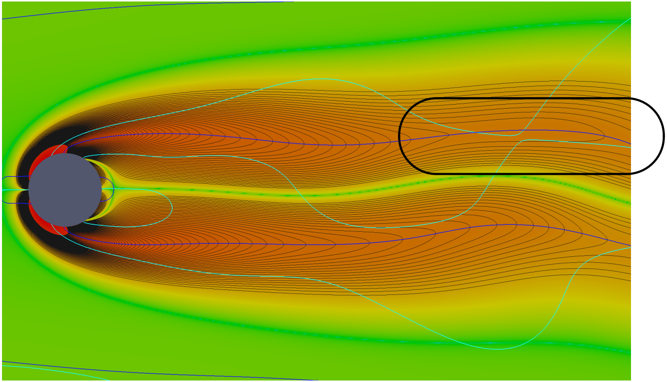

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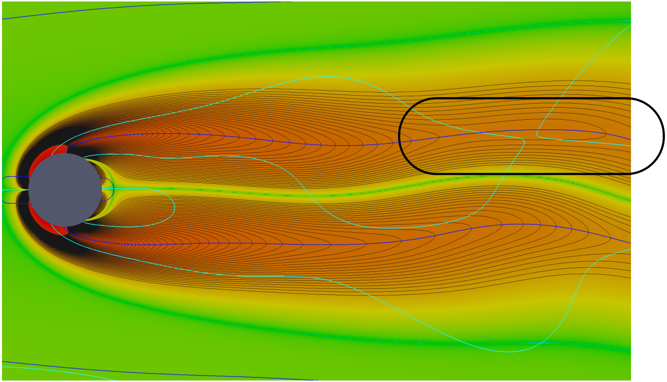

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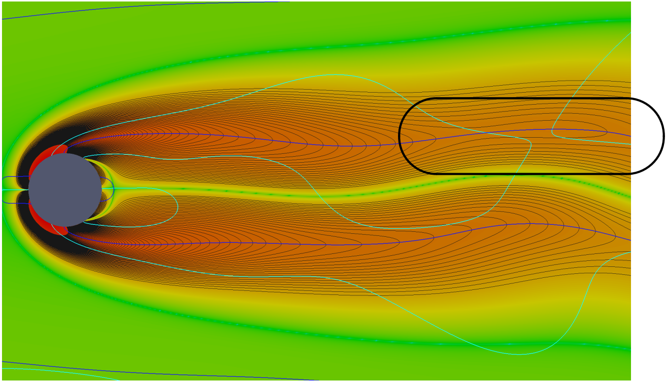

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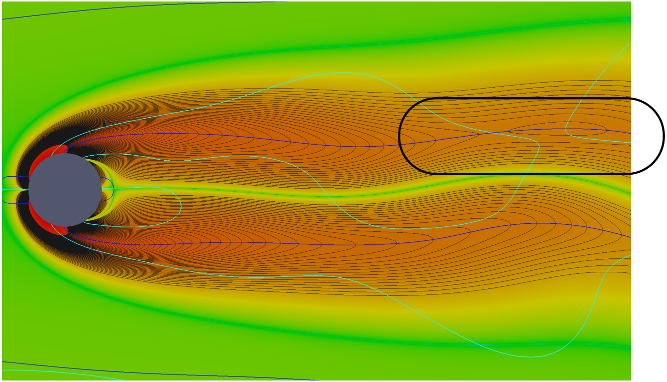

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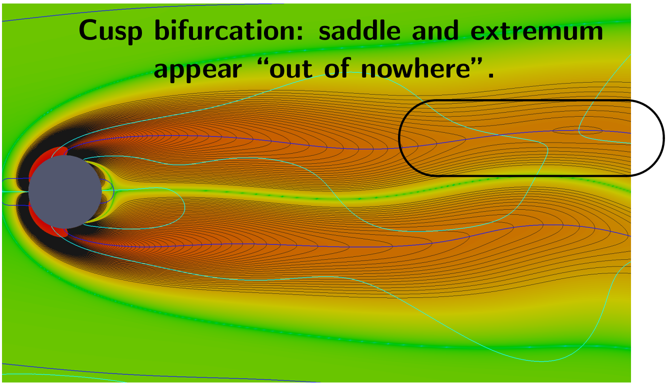

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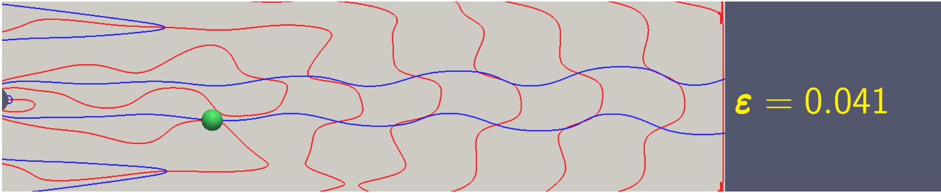

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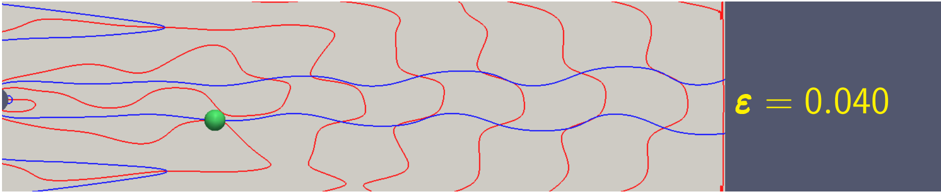

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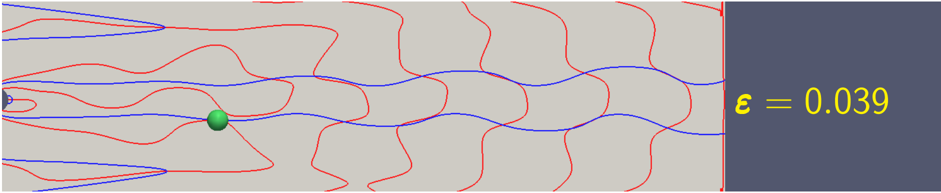

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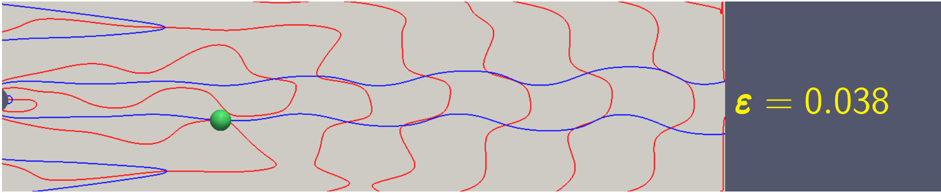

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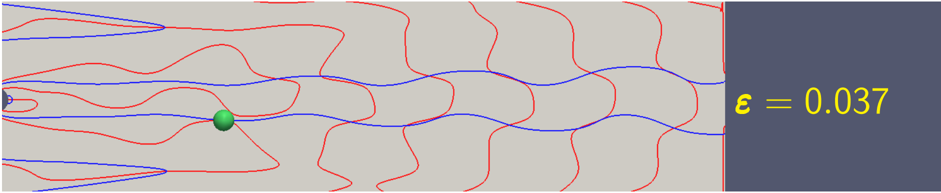

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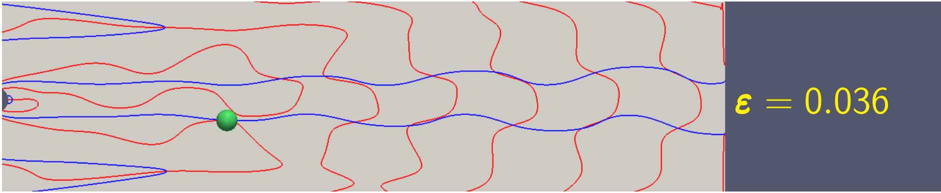

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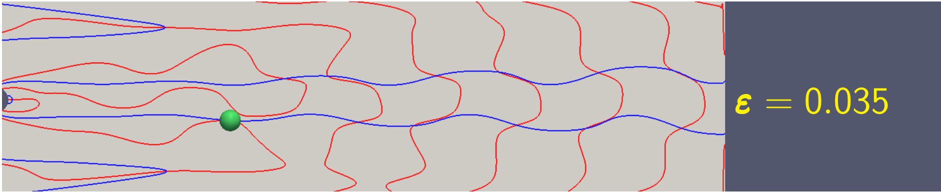

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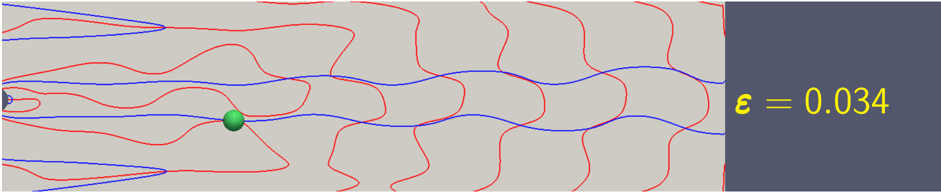

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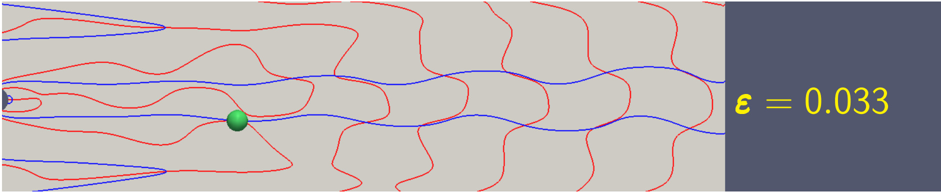

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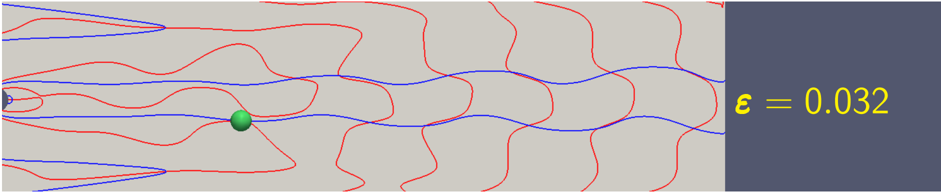

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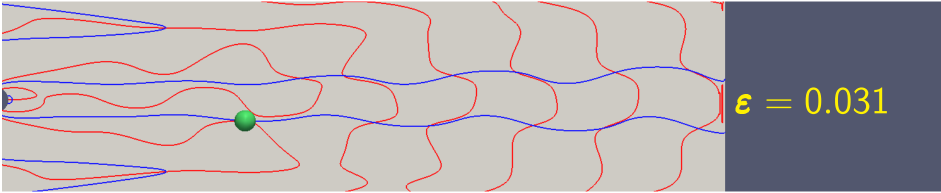

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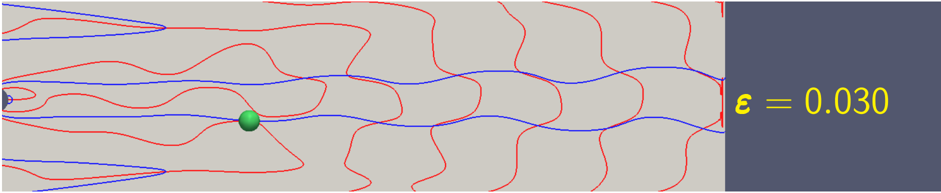

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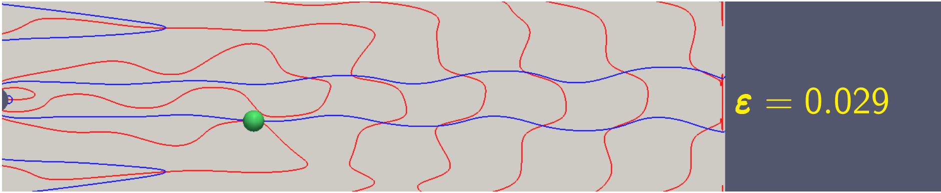

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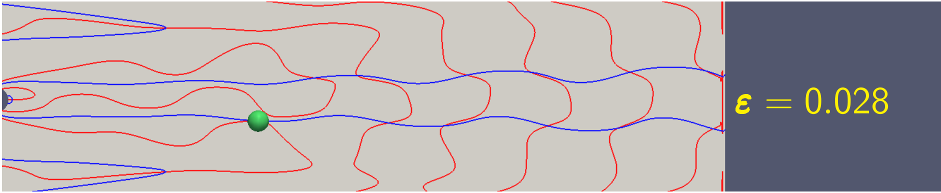

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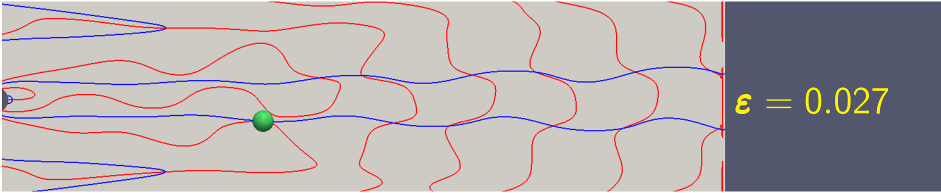

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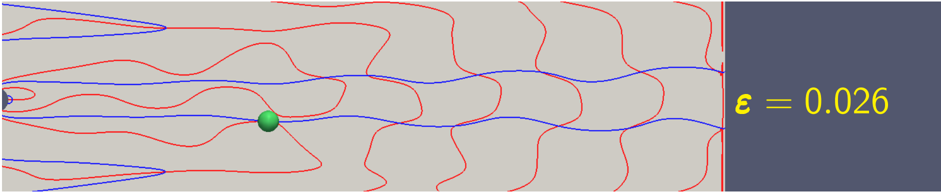

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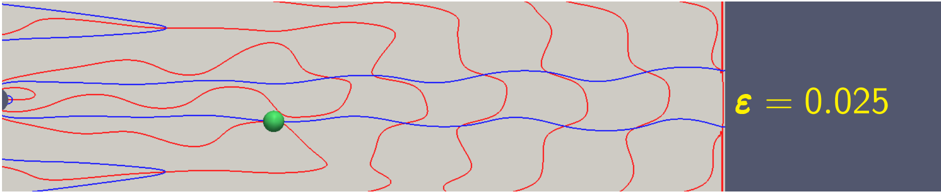

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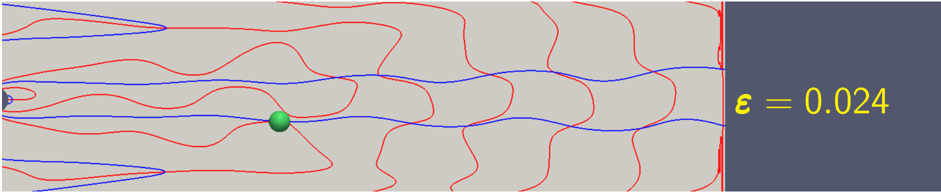

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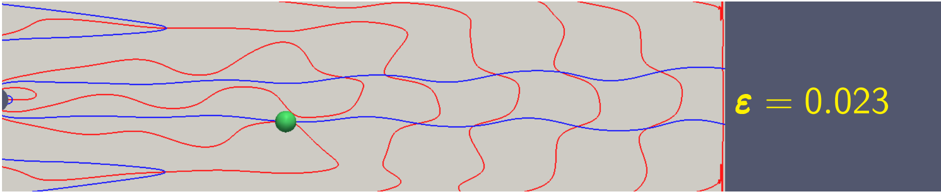

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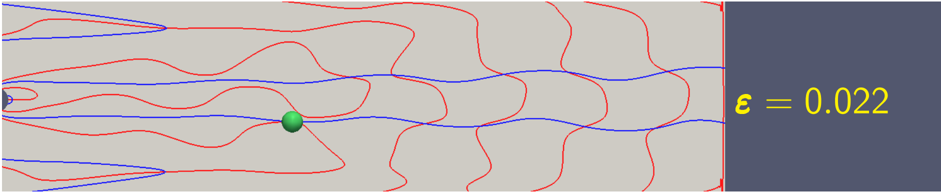

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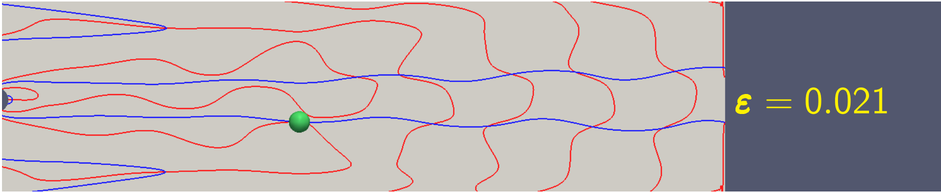

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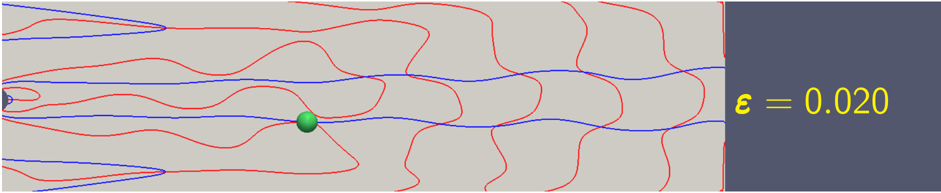

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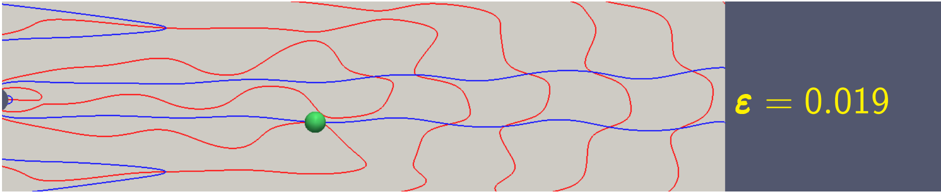

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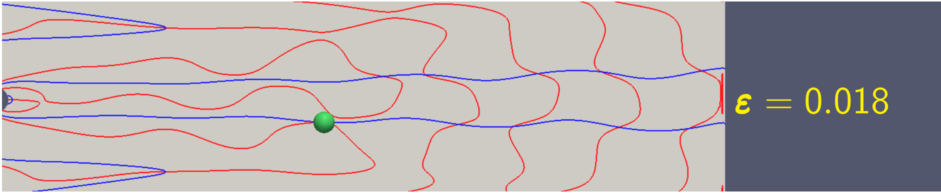

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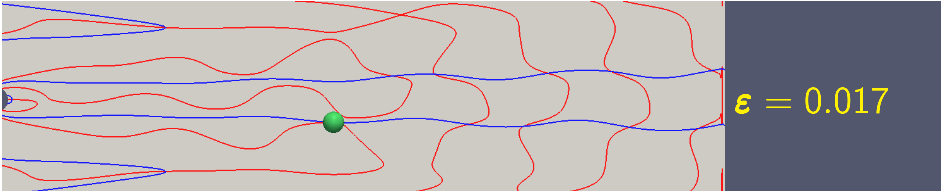

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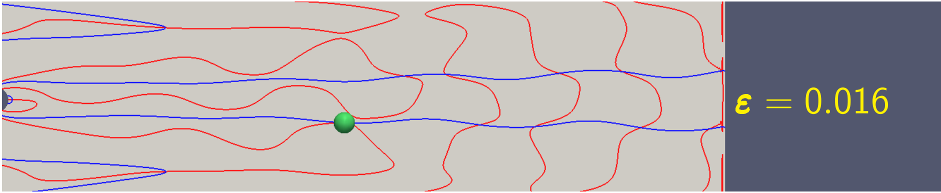

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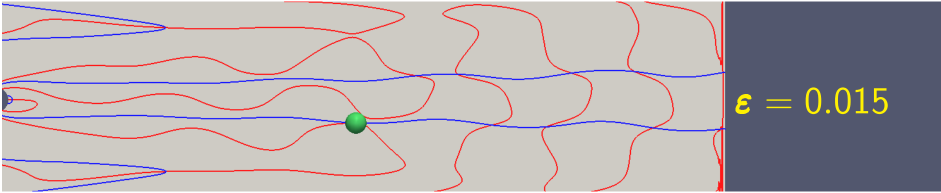

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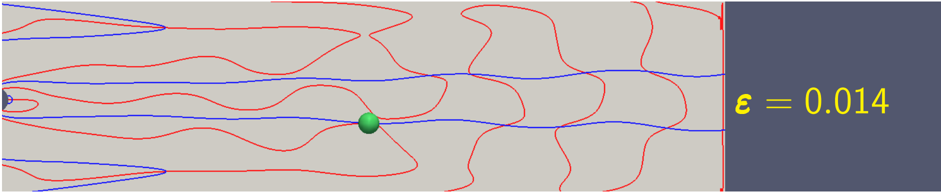

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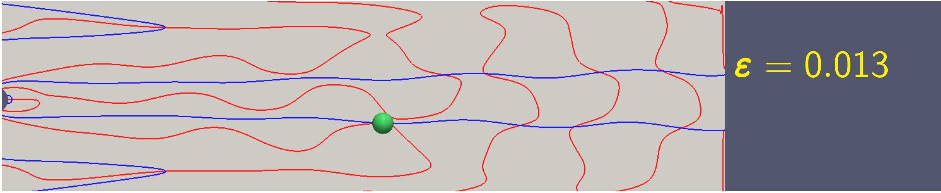

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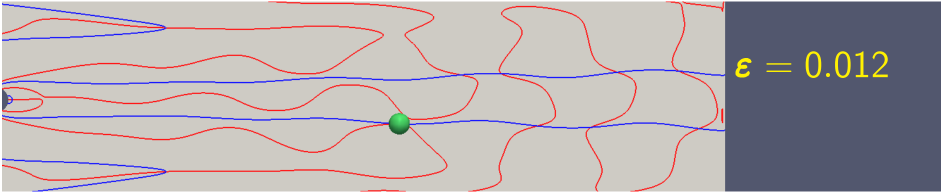
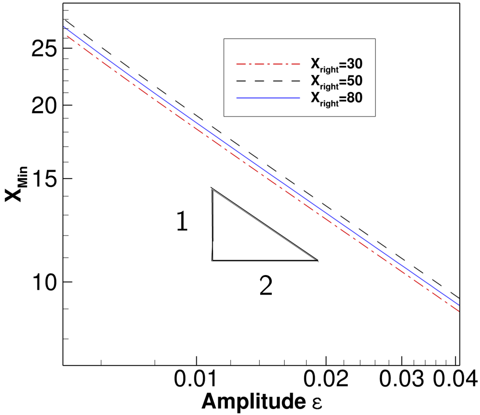


- Cusp doesn't seem to disappear – it just moves downstream as ε is reduced!

Cusp bifurcation "disappears to" infinity as $\varepsilon \to 0$??

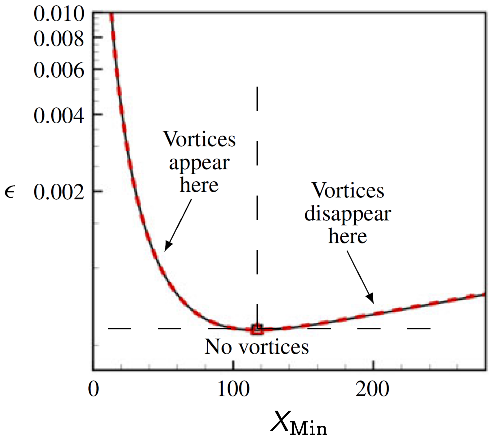
Plot of cusp position, $X_{\mathrm{Min}}(\varepsilon)$:



- **Observation:**

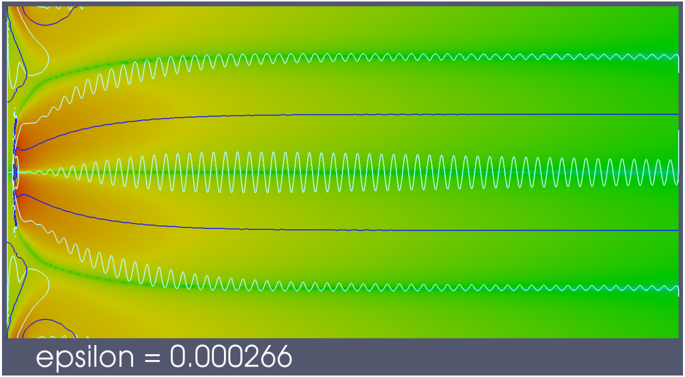
$$X_{\mathrm{Min}} \sim \varepsilon^{-1/2}$$

Well, not quite...



- No vortices are created when $\varepsilon < 0.00057$
- For $\varepsilon = 0.00057$ a vortex is created at $X_{\mathrm{Min}} = 117.1$

So, here's what really happens



- Karman vortex street develops at finite ε.

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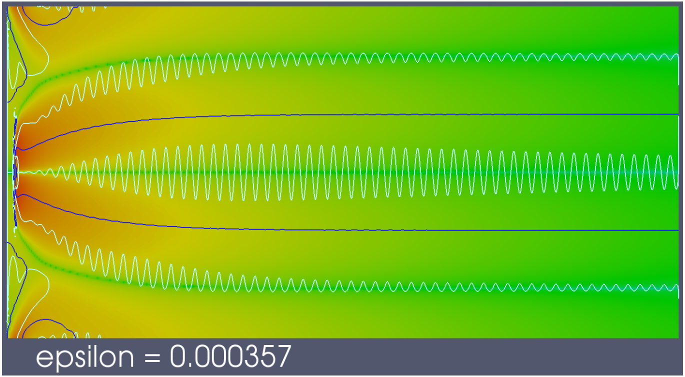

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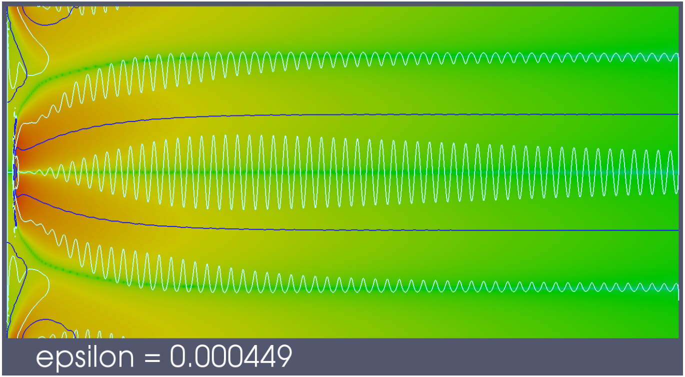

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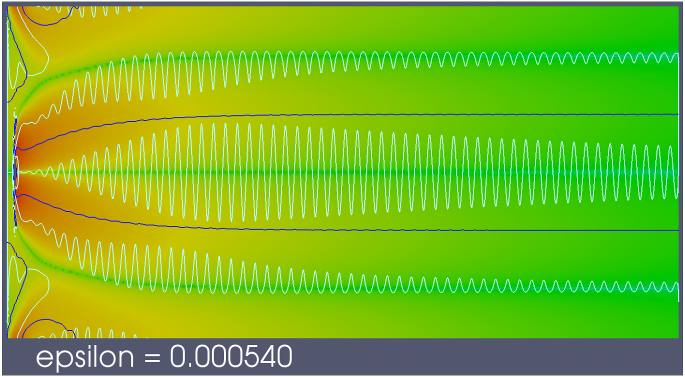

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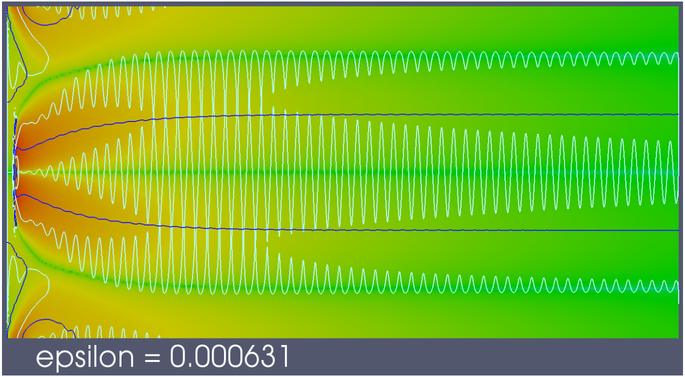

- Karman vortex street develops at finite $0.00054 < \varepsilon < 0.000631$.
- Cusp bifurcation first appears far downstream of cylinder...

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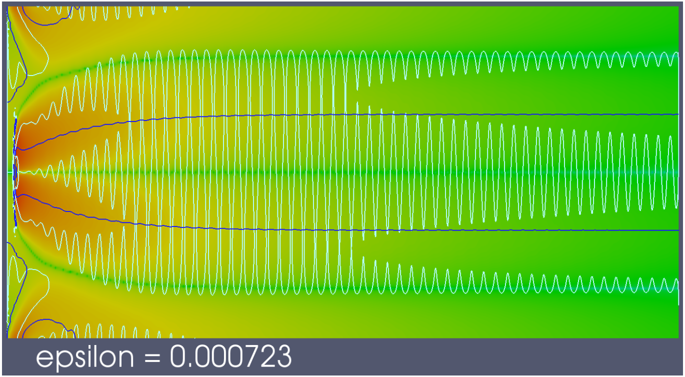

- Karman vortex street develops at finite $0.00054 < \varepsilon < 0.000631$.
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- ...and then moves rapidly upstream as ε is increased.

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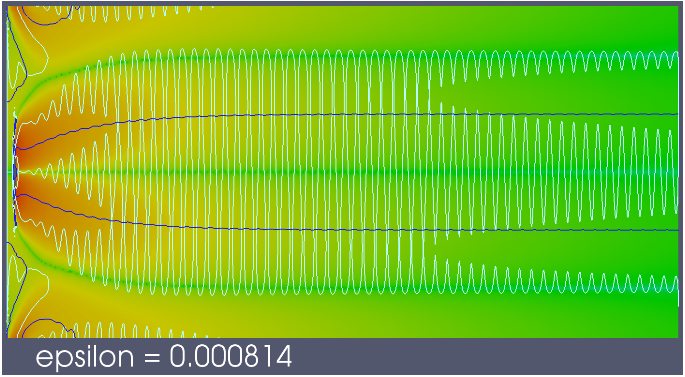
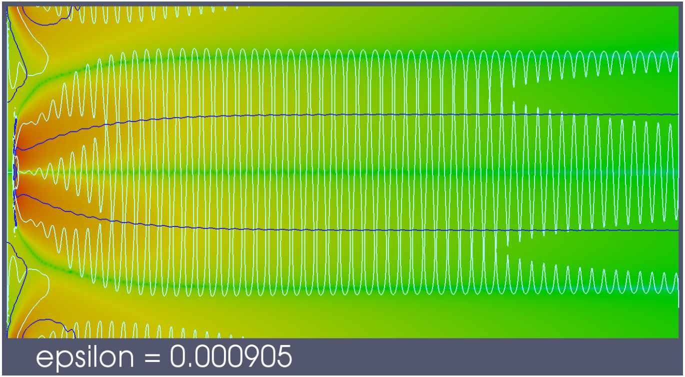


- Karman vortex street develops at finite $0.00054 < \varepsilon < 0.000631$.
- Cusp bifurcation first appears far downstream of cylinder...
- ...and then moves rapidly upstream as ε is increased.
- Karman vortex street persists for finite length and then disappears via a reverse cusp bifurcation when diffusion of vorticity smoothes out the local maxima.

So, here's what really happens



- Karman vortex street develops at finite $0.00054 < \varepsilon < 0.000631$.
- Cusp bifurcation first appears far downstream of cylinder...
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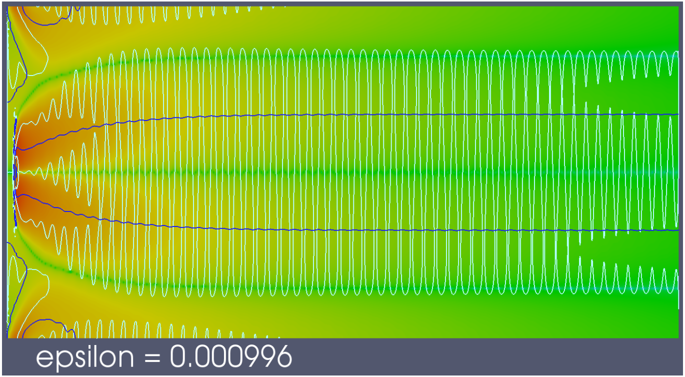

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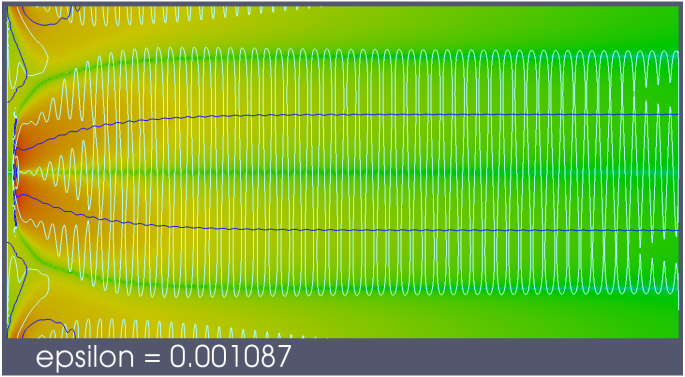

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The bifurcations take place at slightly different Reynolds numbers

Vortex definition # 3:The Q-criterion

The velocity gradient tensor can be decomposed into a symmetric and a skew-symmetric part

$$\nabla \mathbf{v} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \mathbf{S} + \mathbf{\Omega},$$

$$\mathbf{S} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) = \frac{1}{2}\begin{pmatrix} 2\partial_x u & \partial_x v + \partial_y u \\ \partial_x v + \partial_y u & 2\partial_y v \end{pmatrix},$$

$$\mathbf{\Omega} = \frac{1}{2}(\nabla \mathbf{v} - \nabla \mathbf{v}^T) = \frac{1}{2}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}.$$

The Q-criterion: A vortex is a region where rotation dominates shear,

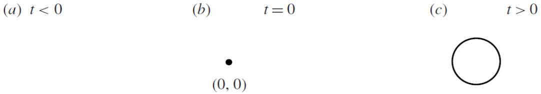
$$Q = ||\mathbf{\Omega}||^2 - ||\mathbf{S}||^2 > 0, \qquad ||\mathbf{A}||^2 = \mathrm{tr}(\mathbf{A}\mathbf{A}^T).$$

$$Q = \det(\nabla \mathbf{v}) = \partial_x u \partial_y v - \partial_y u \partial_x v.$$

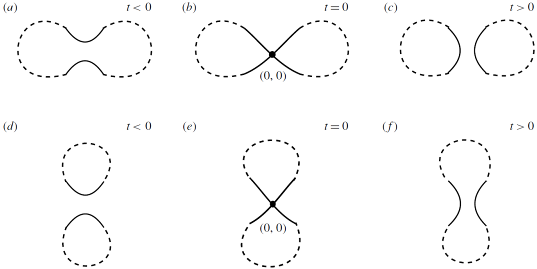
Galilean invariant!

Bifurcation of Q-vortices

Bifurcation occurs at critical points of Q, $\partial_x Q = \partial_y Q = 0$.
If the Hessian of Q is positive or negative definite, and $\partial_t Q \neq 0$, a *punching bifurcation* occurs

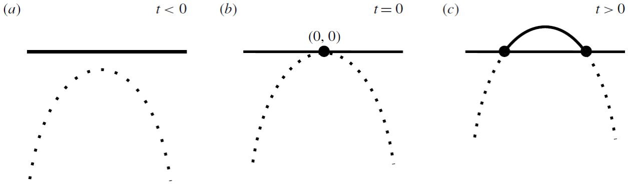


If the Hessian of Q is indefinite, and $\partial_t Q \neq 0$, a *pinching bifurcation* occurs

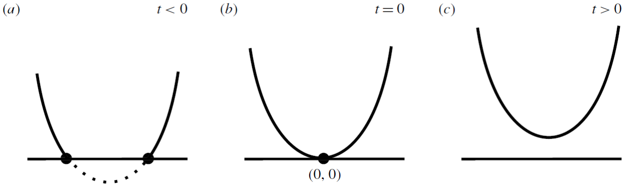


Nielsen, Heil, Andersen, MB, J. Fluid Mech. (2019), vol. 865, pp. 831–849

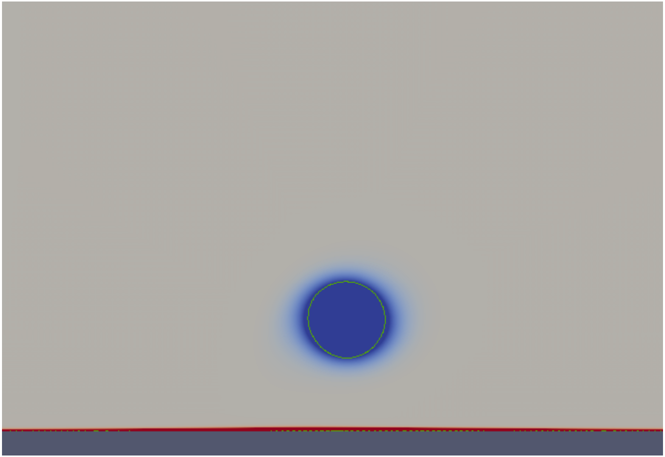
Wall punching



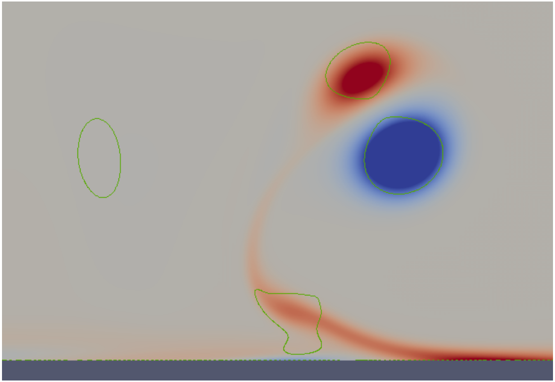
Wall pinching



Q-vortices in boundary layer eruption



Connection between vorticity and Q



There is no simple general connection between critical points of vorticity and Q-vortices. However:

If the flow has rotational symmetry around an extremum of vorticity, there is a Q-vortex around that point.

Summary

- Local extrema of vorticity are a simple generalization of inviscid point vortices to viscous flows
- Equations of motion and a bifurcation theory describing creation and merging of vortices are available
- The vortices in the Karman vortex street are created at a Reynolds number slightly higher than the critical value for onset of oscillations, at a distance ≈ 100 diameters downstream
- The Q-criterion identifies a vortex as a region where vorticity dominates shear
- A bifurcation theory for Q-vortices is available
- What about 3D?
    - Vorticity is a vector field rather than a scalar field. Generalization of critical point not obvious
    - Q is still scalar, so generalization is straightforward